Graph theory preliminary

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In this note, we give a preliminary introduction to graph theory. We will mainly focus on the solutions of discrete Poisson equation, divisors and the complexity. We will closely follow the discussions in Ref. [1, 2].

CONTENTS

I. NOTATIONS AND DEFINITIONS

We briefly introduce the basic concepts and definitions in the graph theory in this section.

Definition I.1. *Γ is a **simple**, **undirected**, and **connected** graph on N vertices, where*

1. *"Simple" means at most one edges between any two vertices and no self-loops.*
2. *"Undirected" means no directed edges.*
3. *"Connected" means any two vertices are connected by a path.*

Definition I.2. *An **adjacency matrix** A is an $\mathsf{N} \times \mathsf{N}$ symmetric matrix given by*

$$\begin{cases} A_{ij} = 1, & \text{if there is an edge } \langle i,j \rangle \text{ between them.} \\ A_{ij} = 0, & \text{otherwise.} \end{cases}$$

Definition I.3. *A **degree matrix** D is an $\mathsf{N} \times \mathsf{N}$ diagonal matrix $D = \operatorname{diag}(d_1, \cdots, d_N)$. The entry d_i is the **degree** of vertex i that is the number of edges incident to i.*

Definition I.4. *A **Laplacian matrix** L is given by $L = D - A$.*

Definition I.5. *A **tree** on N vertices is a graph that contains no cycle or loop. A **spanning tree** of a graph is a tree that includes all of the vertices.*

Now we consider a function $f:\ \Gamma \to X$, where X is an Abelian group. We denote the set of all such functions as $\mathcal{F}(\Gamma, X)$. We can define the *discrete Laplacian operator* $\Delta_L:\ \mathcal{F}(\Gamma, X) \to \mathcal{F}(\Gamma, X)$ as,

$$\Delta_L f(i) = \sum_j L_{ij} f(j) = d_i f(i) - \sum_{j:\langle i,j\rangle \in \Gamma} f(j) = \sum_{j:\langle i,j\rangle \in \Gamma} \left[f(i) - f(j) \right]. \tag{I.1}$$

Consider the *discrete Poisson equation*

$$\Delta_L f(i) = g(i). \tag{I.2}$$

If $g = 0$, then it becomes *discrete Laplace equation*. And now the function $f(i) \in \mathcal{F}(\Gamma, X)$ is said to be *harmonic*. The set of all the discrete harmonic functions is the kernel of Δ_L, i.e. $\ker_X \Delta_L$. Consider the cokernel of Δ_L

$$\mathrm{coker}_X\, \Delta_L = \frac{\mathcal{F}(\Gamma, X)}{\mathrm{Im}_X\, \Delta_L}. \tag{I.3}$$

Then it is natural to understand the solutions of Eq.(I.2). A solution exists if and only if g is in the same equivalence class as 0 in this quotient.

II. SOLUTIONS OF DISCRETE POISSON EQUATIONS

The general solutions of the discrete Poisson equation, if any, can also be found by applying the Smith decomposition of L [3]. It is given by

$$R = PLQ, \quad R_{ab} = \sum_{i,j} P_{ai} L_{ij} Q_{jb}. \tag{II.1}$$

where $P, Q \in GL_{\mathsf{N}}(\mathbb{Z})$ and $R = \mathrm{diag}(r_1, \cdots, r_{\mathsf{N}})$. The diagonal entries are non-negative, known as the *invariant factors*, such that $r_a | r_{a+1}$ for $a = 1, \cdots, \mathsf{N} - 1$. It is essential to discuss the eigenvalues of the Laplacian matrix L. L acts on a N dimensional vector v with each elements being an object on vertex. Writing $\mathbf{v} = (v_1, \cdots, v_{\mathsf{N}})^T$, we have,

$$L\mathbf{v} = \mathbf{w} = (w_1, \cdots, w_{\mathsf{N}})^T \quad \Rightarrow \quad w_i = \deg(i) v_i - \sum_{j:\langle ij\rangle \in \Gamma} v_j = \sum_{j:\langle ij\rangle \in \Gamma} (v_i - v_j). \tag{II.2}$$

Obviously, $\mathbf{v} = (1, \cdots, 1)^T$ is an eigenvector of L with zero eigenvalues. We have the theorem,

Theorem II.1. *For a graph Γ, the number of zero eigenvalues of the Laplacian matrix L is equal to the number of connected components of the graph Γ.*

Proof. Let Γ has k connected components. So Γ can written as a union of disjoint sets, i.e. $\Gamma = S_1 \cup \cdots \cup S_k$. Let us separate the proof into two parts.

- L has at least k zero eigenvalues.

  Define vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ such that

  $$\begin{cases} \mathbf{v}_i(j) = 1, & \text{if } j \in S_i, \\ \mathbf{v}_i(j) = 0, & \text{otherwise,} \end{cases}$$

  where $\mathbf{v}_i(j)$ is the j-th component of the vector $\mathbf{v}_i$. It is easy to check $\mathbf{v}_i$ are eigenvectors with zero eigenvalue of L and any two of them are orthogonal. Hence, $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a set of orthogonal basis of L.

- L has at most k zero eigenvalues.

  Assume we can find another vector $\tilde{\mathbf{v}}$ orthogonal to $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ satisfying $L\tilde{\mathbf{v}} = 0$. Then

  $$\tilde{\mathbf{v}}^T L \tilde{\mathbf{v}} = \sum_{j<j:\langle ij\rangle \in \Gamma} (\tilde{\mathbf{v}}(i) - \tilde{\mathbf{v}}(j))^2 = 0.$$

  It means $\tilde{\mathbf{v}}$ must be constant on every connected vertices. Suppose it is nonzero and constant on all sites in S_i then it cannot be orthogonal to $\mathbf{v}_i$. So there is no way to find the $k+1$-st zero eigenvector.

□

In our case, the graph is fully connected thus only has one component. So L only has one zero eigenvector. Therefore, we have $r_a > 0$ for $a = 1, \cdots, \mathsf{N} - 1$ and $r_{\mathsf{N}} = 0$. Note the summation of entries of L over a row or a column is zero by the definition of L. Using this property, since

$$L_{ij} = \sum_{ab} (P^{-1})_{ia} R_{ab} (Q^{-1})_{bj} = \sum_{a} r_a (P^{-1})_{ia} (Q^{-1})_{aj}, \tag{II.3}$$

$$0 = \sum_i L_{ij} = \sum_a r_a (Q^{-1})_{aj} \sum_i (P^{-1})_{ia} = 0 \quad \Rightarrow \quad \sum_i (P^{-1})_{ia} = 0. \tag{II.4}$$

Similarly,

$$\sum_i (Q^{-1})_{ai} = 0 \tag{II.5}$$

for $a < \mathsf{N}$. So we can choose them to be

$$P^{-1} = \begin{pmatrix} \tilde{P}^{-1} & \mathbf{0} \\ -\mathbf{1}^T \tilde{P}^{-1} & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} \tilde{Q}^{-1} & -\tilde{Q}^{-1}\mathbf{1} \\ \mathbf{0}^T & 1 \end{pmatrix}, \tag{II.6}$$

where $\tilde{P}, \tilde{Q} \in GL_{\mathsf{N}-1}(\mathbb{Z})$. It follows that

$$P = \begin{pmatrix} \tilde{P} & \mathbf{0} \\ \mathbf{1}^T & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} \tilde{Q} & \mathbf{1} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{II.7}$$

Using $RQ^{-1} = PL$, we can define

$$f'_a = \sum_i (Q^{-1})_{ai} f(i), \quad g'_a = \sum_i P_{ai} g(i). \tag{II.8}$$

And the discrete Poisson equation turns to be

$$r_a f'_a = g'_a, \quad a = 1, \cdots, \mathsf{N}. \tag{II.9}$$

Note that we have N independent equations now. Let us first consider the case $a = \mathsf{N}$. The condition for the existence of a solution is $g'_{\mathsf{N}} = \sum_i g(i) = 0$. And f'_{N} can take any value in X which can also be regarded as an zero mode.

Now consider $a < \mathsf{N}$.

1. $X = \mathbb{R}$: We have a unique solution

$$f'_a = \frac{1}{r_a} g'_a \tag{II.10}$$

Then

$$f(i) = \sum_{a<N} \sum_j \frac{Q_{ia} P_{aj}}{r_a} g(j) + c, \tag{II.11}$$

where c is a real constant.

2. $X = U(1)$: There are solutions

$$f'_a = \frac{1}{r_a} g'_a + \frac{2\pi p_a}{r_a}, \tag{II.12}$$

where $p_a = 0, \cdots, r_a - 1$ are series of integers. Then

$$f(i) = \sum_{a<N} \sum_j \frac{Q_{ia} P_{aj}}{r_a} g(j) + 2\pi \sum_{a<N} \frac{Q_{ia} p_a}{r_a} + c, \tag{II.13}$$

where $c \sim c + 2\pi$.

3. $X = \mathbb{Z}$: The solution

$$f'_a = \frac{1}{r_a} g'_a \tag{II.14}$$

exists if and only if r_a divides g'_a. This is equivalent to $g'_a = 0 \mod r_a$ for $a < \mathsf{N}$. Then

$$f(i) = \sum_{a<N} \sum_j \frac{Q_{ia} P_{aj}}{r_a} g(j) + p, \tag{II.15}$$

where p is an integer.

4. $X = \mathbb{Z}_N$: The equation becomes

$$r_a f'_a = g'_a \mod N. \tag{II.16}$$

Let us first review Bézout's identity,

Theorem II.2. *Let a and b be integers. Then there exists integers x and y such that*

$$ax + by = m \gcd(a, b). \tag{II.17}$$

Go back to the original equation. We first divide the Eq.(II.16) by $d = \gcd(r_a, N)$, i.e.

$$\frac{r_a}{d} f'_a = \frac{g'_a}{d} \mod \frac{N}{d}. \tag{II.18}$$

Since $\gcd(r_a/d, N/d) = 1$,

$$\exists \left(\frac{r_a}{d}\right)^{-1} \mod \frac{N}{d}, \quad \text{s.t.} \left(\frac{r_a}{d}\right)^{-1} \frac{r_a}{d} f'_a = \left(\frac{r_a}{d}\right)^{-1} \frac{g'_a}{d} = \frac{g'_a}{r_a} \mod \frac{N}{d}. \tag{II.19}$$

i.e.

$$f'_a = \frac{g'_a}{r_a} \mod \frac{N}{d}. \tag{II.20}$$

To match the notations in Ref. [1], we define $\tilde{r}_a = d/r_a$,

$$f'_a = \frac{g'_a}{d} \tilde{r}_a \mod \frac{N}{d}. \tag{II.21}$$

Explicitly,

$$f'_a = \frac{\tilde{r}_a}{\gcd(N, r_a)} g'_a + \frac{N p_a}{\gcd(N, r_a)} \tag{II.22}$$

where $p_a \sim p_a + \gcd(r_a, N)$. So there are $\gcd(N, r_a)$ inequivalent solutions. The solution of this form exists if and only if $\gcd(N, r_a) | g'_a$. Then,

$$f(i) = \sum_{a<\mathsf{N}} \sum_j \frac{Q_{ia} \tilde{r}_a P_{aj}}{\gcd(N, r_a)} g(j) + \sum_{a<\mathsf{N}} \frac{N Q_{ia} p_a}{\gcd(N, r_a)} + p, \tag{II.23}$$

where p is an integer modulo N. We can also absorb the integer and write a more compact expression. Notice that $\gcd(r_\mathsf{N}, N) = N$ and $Q_{i\mathsf{N}} = 1$. We define $p_\mathsf{N} = p \mod N$, then general solution is

$$f(i) = \sum_{a<\mathsf{N}} \sum_j \frac{Q_{ia} \tilde{r}_a P_{aj}}{\gcd(N, r_a)} g(j) + \sum_{a=1}^{\mathsf{N}} \frac{N Q_{ia} p_a}{\gcd(N, r_a)}. \tag{II.24}$$

When $g = 0$, we have $\mathbb{Z}_N$-valued discrete harmonic function.

III. JACOBIAN GROUP AND COMPLEXITY

We will introduce the Jacobian group of a graph in this part which relates to the complexity. Let us set X to be an integer set $\mathbb{Z}$. The function $f:\Gamma\to\mathbb{Z}$ in the set $\mathcal{F}(\Gamma,\mathbb{Z})$ assigns an integer to every vertex in the graph. Now the Laplacian operator $\Delta_L:\mathcal{F}(\Gamma,\mathbb{Z})\to\mathcal{F}(\Gamma,\mathbb{Z})$. Therefore we have the following definitions.

Definition III.1. *A **divisor** is an element q in $\mathcal{F}(\Gamma,\mathbb{Z})$ such that*

$$q:\Gamma\to\mathbb{Z}.$$

*We name $\mathcal{F}(\Gamma,\mathbb{Z})$ as $\mathrm{Div}(\Gamma)$. Given a divisor, its **degree** is defined as $\deg q=\sum_i q(i)$. $\mathrm{Div}^k(\Gamma,\mathbb{Z})$ denotes the set of all k-degree divisors.*

Definition III.2. *An element of $\mathrm{Im}_{\mathbb{Z}}\,\Delta_L$ is known as the **principal divisor**.*

By definition, any principal divisor has degree zero. They form the group

$$\mathrm{Div}^0(\Gamma)=\{q\in\mathrm{Div}(\Gamma)|\deg(q)=0\} \tag{III.1}$$

Definition III.3. *The **Picard group** and **Jacobian group** is defined as*

$$\mathrm{Pic}(\Gamma)=\frac{\mathrm{Div}(\Gamma,\mathbb{Z})}{\mathrm{Im}_{\mathbb{Z}}\,\Delta_L}=\mathrm{coker}_{\mathbb{Z}}(\Delta_L),\quad \mathrm{Jac}(\Gamma)=\frac{\mathrm{Div}^0(\Gamma,\mathbb{Z})}{\mathrm{Im}_{\mathbb{Z}}\,\Delta_L}. \tag{III.2}$$

There exists an isomorphism

$$\mathrm{Jac}(\Gamma)\cong\prod_{a<\mathsf{N}}\mathbb{Z}_{r_a}. \tag{III.3}$$

This can be seen as follow. For all vertices $(v_1,\cdots,v_{\mathsf{N}})$ in the graph, the images of q form a vector $(q(1),\ dots,q(\mathsf{N}))\in\mathbb{Z}^{\mathsf{N}}$. Thus,

$$\mathrm{Div}(\Gamma)\cong\mathbb{Z}^{\mathsf{N}}. \tag{III.4}$$

For $\mathrm{Div}^0(\Gamma)$, the constraint $q(1)+\cdots+q(\mathsf{N})=0$ is imposed. it leads to only $\mathsf{N}-1$ independent integers. Hence,

$$\mathrm{Div}^0(\Gamma)\cong\mathbb{Z}^{\mathsf{N}-1}. \tag{III.5}$$

Consider the image of the Laplacian operator,

$$\mathrm{Im}_{\mathbb{Z}}\,\Delta_L=\Delta_L f(i)=L\mathbf{f}, \tag{III.6}$$

where $\mathbf{f}=(f(1),\cdots,f(\mathsf{N}))^T\in\mathbb{Z}^{\mathsf{N}}$. Under the Smith decomposition, $R=PLQ=\mathrm{diag}(r_1,\cdots,r_{\mathsf{N}-1}$, we have the isomorphism,

$$\mathrm{Im}_{\mathbb{Z}}\,\Delta_L\cong r_1\mathbb{Z}\times\cdots\times r_{\mathsf{N}-1}\mathbb{Z}. \tag{III.7}$$

Finally,

$$\mathrm{Jac}(\Gamma)=\frac{\mathrm{Div}^0(\Gamma,\mathbb{Z})}{\mathrm{Im}_{\mathbb{Z}}\,\Delta_L}\cong\prod_{a<\mathsf{N}}\mathbb{Z}_{r_a}.$$

Furthermore, it is also easy to check

$$\mathrm{Pic}(\Gamma)\cong\mathbb{Z}\times\mathrm{Jac}(\Gamma). \tag{III.8}$$

The order of $\mathrm{Jac}(\Gamma)$ can be expressed in terms of invariants r_i or eigenvalues of L:

$$|\mathrm{Jac}(\Gamma)|=\prod_{a<\mathsf{N}}r_a=\frac{\lambda_2\cdots\lambda_{\mathsf{N}}}{\mathsf{N}}, \tag{III.9}$$

where $0 = \lambda_1 < \lambda_2 \leq \cdots \lambda_{\mathsf{N}}$ are eigenvalues of L. By Kirchhoff's matrix-tree theorem [4, 5], it is equal to the number of spanning trees of Γ. A tree is an undirected connected graph with no cycles. It is a spanning tree of Γ if it includes all vertices of Γ. The spanning tree measures how connected a graph is and is the well-known notion complexity of a graph.

Lastly, it is easy to generalize $\mathbb{Z}$ to $\mathbb{Z}_N$. Then the mod-N reduction of Jacobian group $\mathrm{Jac}(\Gamma, N)$ has the isomorphism

$$\mathrm{Jac}(\Gamma, N) \cong \prod_{a=1}^{\mathsf{N}} \mathbb{Z}_{\gcd(r_a, N)}. \tag{III.10}$$

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